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Global Approximation with Bounded Coefficients

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1. INTRODUCTION

Let X be a normed linear space, G a subset of X, and f an element of X. Then, a best approximant g_* to f from G (if it exists) is a solution to

$$\| g_* - f \| = \inf_{g \in G} \| g - f \|. \tag{1.1}$$

Geometrically, g_* is the center of a ball containing f with the smallest radius among all balls that contain f with centers in G. It may happen that f is not known exactly, but is known to lie in a bounded set F. It is reasonable, then, to approximate simultaneously all $f \in F$ by solving

$$\sup_{f \in F} \| g_* - f \| = \inf_{g \in G} \sup_{f \in F} \| g - f \| = r_G(F), \tag{1.2}$$

where the number $r_G(F)$ is the radius of F with respect to G.

We call the set of solutions, which may be empty, $E_G(F)$. Any $g_* \in E_G(F)$ is the center of a ball that contains F with smallest radius among all balls containing F with centers in G. Thus, we view problem (1.2) as a natural generalization of the approximation problem (1.1) and in fact $E_G(\{w\})$, for singletons, is the set of best approximations to f from G.

If $G = X$, then the solutions of (1.2) are called Chebyshev centers, following Garkavi. The problems of existence, uniqueness, and characterization of Chebyshev centers have been studied in [4, 5, 8]. For an accessible survey of these results, see [7]. With this in mind, we call solutions to (1.2) restricted Chebyshev centers (or restricted centers) of F with respect to G.

In Sections 2 and 3, general results concerning the existence and uniqueness of restricted centers are presented. The major results are contained in Sections 5 and 6, where X is the set of continuous functions on a compact interval with either the supremum norm or the L^1 norm and G is a subset of a finite-dimensional subspace determined by coefficient restrictions. In general, the thrust of theorems 5.1 and 6.1, is that $E_G(F)$ is a singleton when $E_X(F) \cap G = \varnothing$. These results rest on the work of Laurent and Tuan [10], who have provided a general framework within which to pose these problems. Section 4 adapts the work of Laurent and Tuan for application in later sections. We were led to the consideration of restricted coefficients by the paper of Roulier and Taylor [12].

Throughtout this work, we adopt the following notation. For any normed linear space X, $S(X)$ denotes the unit sphere in X, i.e., $S(X) = \{x \in X: \| x \| = 1\}$ We use Θ as the zero element of a generic vector space. For any set $A \subset X$ we denote the closed convex hull of A by $\overline{\text{co}}\, A$, and the extreme points of a convex set C by $\text{Ext}(C)$.

2. Existence of Restricted Centers

In this section, we present a few general results on the existence of restricted centers from closed subsets of a normed linear space. As might be expected, the basic results are similar to the results on existence of best approximations.

THEOREM 2.1. *If G is a weak-star closed subset of a dual space X^*, then $E_G(F) \neq \varnothing$ for every bounded set F in X^*.*

Proof. Let F be a bounded subset of X^*. Then for each $n > 0$, there exists a $g_n \in G$ satisfying

$$\sup_{f \in F} \| f - g_n \| \leqslant r_G(F) + (1/n). \tag{2.1}$$

Now, $\{g_n\}$ is a bounded subset of G and hence, has a weak-star convergent subsequence (which we will also call g_n) converging to a $g_0 \in G$. Since the norm is weak-star sequentially-lower-semicontinuous, we know, for every $f \in F$,

$$\| f - g_0 \| \leqslant \liminf \| f - g_n \| \leqslant r_G(F). \tag{2.2}$$

Therefore, $g_0 \in E_G(F)$.

COROLLARY 2.1. *If V is a reflexive subspace of a normed linear space X*

and G is a closed convex subset of V, then $E_G(F) \neq \varnothing$ *for every bounded* $F \subset X$.

Proof. Embed X in X^{**} and note that G is weak-star closed.

3. Uniqueness in Rotund Spaces

The results of this section are straightforward extensions of results that can be found in [7]; the first and third are due (independently) to Klee and Garkavi, the second to Smith, the second-named author of this paper. A somewhat unusual condition is known ("uniformly rotund in every direction") that is necessary and sufficient for every bounded subset of X to have at most one Chebyshev center [4]. This condition is strictly stronger than rotundity and strictly weaker than uniform rotundity. For most practical purposes, however, the following results are adequate.

THEOREM 3.1. *Let X be a uniformly rotund Banach space. Suppose G is a closed convex subset of X and F is a bounded subset of X. Then* $E_G(F)$ *is a singleton.*

Proof. We know by Theorem 2.1 that $E_G(F)$ is not empty. The remainder of the proof is essentially contained in [7, pp. 187–188].

The next result is a straightforward extension of a result proved by Smith [7, p. 188]. We define an E-space (for existence) as a rotund Banach space X such that every weakly closed set C in X is approximatively compact (i.e., C satisfies the following condition: If

$$\{x_n\} \subset C \qquad \text{and} \qquad \| x - x_n \| \to \min_{c \in C} \| x - c \|,$$

then $\{x_n\}$ has a strongly converging subsequence). It is known that X is an E-space if and only if X is reflexive, rotund, and if $\{x_n\}$ and x are in $S(X)$ and x_n converges weakly to x, then x_n converges strongly to x.

It is possible to weaken the hypothesis in Theorem 3.1 of uniform rotundity to rotundity if we consider only compact sets F.

THEOREM 3.2. *If X is rotund and G is a closed convex subset of X, then for every compact set F,* $E_G(F)$ *contains at most one point. Further, if X is an E-space and F is as above, then* $E_G(F)$ *is a singleton and* $E_G(\ )$ *is continuous with respect to the Hausdorff metric on the compact subsets of X.*

A proof of this theorem can easily be constructed from that given in [7, p. 188]. Finally, we present the following interesting characterization of a Hilbert space ($\dim X \geqslant 2$).

THEOREM 3.3. *For a normed linear space X the following are equivalent.*

(A) *For each bounded $F \subset X$ and each closed subspace $M \subset X$*

$$E_M(F) \cap \overline{\mathrm{co}}\left(\bigcup_{f \in F} E_M(f)\right) \neq \varnothing. \tag{3.1}$$

(B) *Dimension of $X \leqslant 2$ or X is a Hilbert space.*

Proof. Recall that $E_M(f)$ is the best approximation operator when $\{f\}$ is a singleton. We first prove that $(B) \Rightarrow (A)$ when X is a Hilbert space. We know by Theorem 3.1 that $E_M(F)$ is a singleton $\{x_0\}$. Let $K = \overline{\mathrm{co}}(\bigcup_{f \in F} E_M(f))$ and suppose $x_0 \notin K$, contrary to (3.1). Let H be a hyperplane in M strictly separating x_0 and K (without loss of generality we assume that $\Theta \in H$). Let P_H denote the orthogonal projection onto H and let $x = P_H(x_0)$. Since H strictly separates K and x_0, we have $x \neq x_0$. We now show that $x \in E_M(F)$. For any $k \in K$ let $z \in H$ be on the line segment between x_0 and k. Then,

$$\begin{aligned} \| k - x \| &\leqslant \| k - z \| + \| z - x \| \\ &= \| k - z \| + \| z - P_H(x_0) \| \\ &= \| k - z \| + \| P_H(z - x_0) \| \\ &\leqslant \| k - z \| + \| z - x_0 \| = \| k - x_0 \|. \end{aligned} \tag{3.2}$$

The last inequality follows since P_H is a norm-one projection. Now, for any $f \in F$ with $E_M(f) = \{k\}$, we use the Pythagorem theorem to show

$$\begin{aligned} \| f - x \|^2 &= \| f - k \|^2 + \| k - x \|^2 \\ &\leqslant \| f - k \|^2 + \| k - x_0 \|^2 = \| f - x_0 \|^2. \end{aligned} \tag{3.3}$$

Thus, it follows that x is closer to every $f \in F$ than x_0, so that x and x_0 must both be in $E_M(F)$. The uniqueness result of Theorem 3.1 then implies that $x_0 = x$, contrary to our assumptions. Thus, (B) implies (A) if X is a Hilbert Space. However, if dim X is one or two and M is X, then, for any one-dimensional subspace H, there always exists a norm-one projection onto H corresponding to P_H in (3.2) and the result follows as in [5]. If dim $X = 2$ and $\dim(M) = 1$, then the convexity of the distance functional shows that $E_M(F) \cap \overline{\mathrm{co}}(\bigcup_{f \in F} E_M(f)) \neq \varnothing$. For (A) implies (B), let $M = X$ and apply the theorem of Klee and Garkavi [7, p. 190].

Remark 1. In (A), one may replace "bounded F" by "three point set F," see [5].

Remark 2. Even if X is a Hilbert space, (B) does not hold for all convex sets M. Consider $X = R^2$ with the Euclidean norm and let M be the unit ball. Then, for any two points x_1 and x_2 that are linearly independent and $\text{co}\{x_1, x_2\} \cap M = \varnothing$, it follows that

$$E_M\{x_1, x_2\} \not\subset \overline{\text{co}}\,\{E_M(x_1) \cup E_M(x_2)\}.$$

4. Characterization

We now confront the problem of characterizing restricted centers from finite-diminsional convex subsets G. The basic results in this section are due to Laurent and Tuan [10]. Following [10], we let V be an m-diminsional subspace of a normed linear space X and W a translate of V. Suppose that H is a weak-star compact subset of X^*, the continuous dual of X and that w is a weak-star continuous functional on H. Define the convex functional $c: X \to R$ by

$$c(x) = \sup_{h \in H} [h(x) - w(h)] \tag{4.1}$$

and set $C = \{x: c(x) \leqslant 0\}$. Let $G = C \cap W$. In addition, we make the assumption

$$\{g \in W: c(g) < 0\} \neq \varnothing. \tag{4.2}$$

With these definitions, we can now state a theorem that characterizes elements of $E_G(F)$ for a compact $F \subset X$. This theorem is a consequence of [10, Theorem 2.1].

THEOREM 4.1. *Assume* (4.2). *Let F be a compact set and let G be as above. Then, there exist*

$$\begin{aligned} &\{\phi_1, \ldots, \phi_r\} \subset \text{Ext}(S(X^*)), \\ &\{f_1, \ldots, f_r\} \subset F, \\ &\{h_1, \ldots, h_s\} \subset H, \qquad r \geqslant 1,\ s \geqslant 0,\ r + s \leqslant m + 1, \end{aligned} \tag{4.3}$$

and positive constants $\lambda_1, \ldots, \lambda_r$, $\mu_1, \ldots, \mu_s$ satisfying

$$\sum_{i=1}^{r} \lambda_i \phi_i(x) + \sum_{i=1}^{s} \mu_i h_i(x) = 0, \qquad \textit{for all} \quad x \in V \tag{4.4}$$

so that $g \in G$ is in $E_G(F)$ if and only if we have

$$\begin{aligned} &\phi_i(g - f_i) = \| g - f_i \| = \max_{f \in F} \| g - f \|, \qquad i = 1, \ldots, r, \\ &h_i(g) = w(h_i), \qquad i = 1, \ldots, s. \end{aligned} \tag{4.5}$$

This theorem differs from [10, Theorem 2.1] in two aspects. In the latter, $g \in G$ is in $E_G(F)$ if and only if there exist elements, possibly depending on g, satisfying (4.3)–(4.5). However, Theorem 4.1, asserts the existence of these elements independent of the particular solution g. Secondly, a more technical difference involves choosing the ϕ_i from $\text{Ext}(S(X^*))$ rather than $S(X^*)$. This second point is demonstrated in [10, Theorem 2.2]. Thus, all that remains is to show that the elements in (4.3) may be chosen independent of the particular solution. If $E_G(F)$ is a singleton, we are done by the above remarks. Let g_* be in the relative interior of $E_G(F)$. Applying [10, Theorem 2.1], we obtain sets (4.3) satisfying (4.4) and (4.5) for $g = g_*$. We claim that the sets in (4.3) obtained for g_* actually characterize all solutions. If $g_1 \in G$ satisfies (4.5), then $g_1 \in E_G(F)$ by [10, Theorem 2.1]. Conversely, let g_1 be in $E_G(F)$ with $g_1 \neq g_*$. Then, there is a $g_2 \neq g_*$ in $E_G(F)$ so that $g_* = \alpha g_1 + \beta g_2$ with $\alpha + \beta = 1$ and $\alpha, \beta > 0$, since g_m is in the relative interior of $E_G(F)$ .It is easy to see that $\phi_i(g_k - f_i) = \phi_i(g_* - f_i)$ for $i = 1,\ldots, r$ and $k = 1, 2$ since

$$\begin{aligned} \phi_i(g_* - f_i) &= \| g_* - f_i \| \geqslant \| g_k - f_i \| \\ &\geqslant \phi_i(g_k - f_i). \end{aligned} \tag{4.6}$$

The first inequality follows from the fact that $g_k \in E_G(F)$. Thus, if we had strict inequality for some $k = 1, 2$ and some $i = 1,\ldots, r$ we would have

$$\begin{aligned} \phi_i(g_* - f_i) &= \phi_i(\alpha g_1 + \beta g_2 - f_i) \\ &= \alpha\phi_i(g_1 - f_i) + \beta\phi_i(g_2 - f_i) \\ &< \alpha\phi_i(g_* - f_i) + \beta\phi_i(g_* - f_i) \\ &= \phi_i(g_* - f_i), \end{aligned} \tag{4.7}$$

which of course is false. Similarly, we must have $h_i(g_k) = w(h_i)$ for $i = 1,\ldots, s$ and $k = 1, 2$. For $g_k \in G$ implies that $c(g_k) \leqslant 0$, which, by (4.1), implies that $h_i(g_k) - w(h_i) \leqslant 0$. Thus,

$$h_i(g_k) \leqslant w(h_i) = h_i(g_*). \tag{4.8}$$

Arguing as in (4.7) with ϕ_i replaced by h_i yields the result.

We have shown that (4.5) holds for any $g_1 \in E_G(F)$. This completes the proof.

5. Restricted Coefficients in $C[a, b]$

In the next two sections, we consider the problem of determining whether $E_G(F)$ is a singleton. We assume throughout that G has the following structure:

Let $v_1, \ldots, v_n$ be linearly independent elements of X and set

$$G = \left\{ \sum_{i=1}^{n} c_i v_i : l_i \leqslant c_i \leqslant u_i , i = 1, \ldots, n \right\}. \tag{5.1}$$

To avoid trivialities, we assume that:

$$\begin{aligned} &\text{(i)} \quad u_i \text{ may be } +\infty, \text{ but not } -\infty; \\ &\text{(ii)} \quad l_i \text{ may be } -\infty, \text{ but not } +\infty; \\ &\text{(iii)} \quad l_i \leqslant u_i . \end{aligned} \tag{5.2}$$

Set $I_1 = \{i : l_i = u_i\}$, $I_2 = \{i : l_i \neq u_i$, not both extended reals$\}$ and $I_3 = \{1, \ldots, n\} \backslash (I_1 \cup I_2)$.

Since we will apply Theorem 4.1 to G, we now show that G satisfies the hypothesis in that theorem under the provision that $I_2 \cup I_3 \neq \varnothing$. In the notation of Section 4,

$$V = \left\{ \sum_{i \in I_2 \cup I_3} c_i v_i \right\}, \qquad W = V + \sum_{i \in I_1} l_i v_i . \tag{5.3}$$

To obtain H, we note that $\psi_j \in V^*$ defined by

$$\psi_j(v) = \psi_j \left\{ \sum_{i \in I_1 \cup I_2} c_i v_i \right\} = c_j , \qquad j \in I_2 \tag{5.4}$$

has an extension $\bar{\psi}_j$ to X^*. Set

$$\begin{aligned} H_1 &= \{\bar{\psi}_j : u_j < \infty\}, \\ H_2 &= \{-\bar{\psi}_j : l_j > -\infty\}, \\ H &= H_1 \cup H_2 . \end{aligned} \tag{5.5}$$

We can now define the map $W: H \to R$ by

$$\begin{aligned} w(h) &= u_i , \qquad \text{if} \quad h = \bar{\psi}_i \in H_1 , \\ &= -l_i , \qquad \text{if} \quad h = -\bar{\psi}_i \in H_2 , \end{aligned} \tag{5.6}$$

and define c as in (4.1). If $I_2 = \varnothing$ by convention we will define c to be identically -1. Now, it is not hard to see that G as in (5.1) is $C \cap W$, where as before $C = \{x : c(x) \leqslant 0\}$. We further note that (4.2) is satisfied since $I_2 \cup I_3 \neq \varnothing$.

We now let $X = C[a, b]$ with the supremum norm. A set of continuous functions on $[a, b]$, $\{x_i\}_{i=1}^{n}$ or its span is called Haar if $x = \sum_{i=1}^{n} \alpha_i x_i$ has n zeros implies that x is the zero function. Now we may state the following theorem.

THEOREM 5.1. *Let $X = C[a, b]$ and F be a compact subset of X. Let G be defined as in* (5.1) *and suppose that for every $J \subset I_2$*

$$\{v_i\}_{i \in J \cup I_3} \text{ is a Haar system.} \tag{5.7}$$

If $E_X(F) \cap G = \varnothing$, then $E_G(F)$ is a singleton.

Proof. Recalling that the extreme points of $S(C[a, b]^*)$ are plus or minus, the point evaluation functionals, Theorem 4.1 asserts the existence of r functionals and associated r elements of F and s elements of H satisfying (4.3)–(4.5). In particular, let

$$J_1 = \{j_i: h_i = \pm\bar{\psi}_{j_i}, i = 1,..., s\}. \tag{5.8}$$

Then, for any two elements g_1 and g_2 in $E_G(F)$, we must have (by the second line of (4.5) and (5.8))

$$g_1 - g_2 = \sum_{i \in L} \alpha_i v_i, \qquad L = (I_2 \backslash J_1) \cup I_3, \tag{5.9}$$

Since $E_X(F) \cap G = \varnothing$, it follows that

$$\sum_{i=1}^{r} \lambda_i \phi_i \neq \Theta. \tag{5.10}$$

Indeed, if $\sum_{i=1}^{r} \lambda_i \phi_i = \theta$, then any $g \in E_G(F)$ is in $E_X(F)$ by [10, Theorem 1.1]. Let $M = \text{span}\{v_i : i \in L\}$. Then by (4.5)

$$r + s \leqslant \text{card}(I_2 \cup I_3) + 1 \tag{5.11}$$

hence

$$r \leqslant \text{card}(L) + 1 = \dim M + 1.$$

On the other hand, $\sum_{i=1}^{r} \lambda_i \phi_i$ annihilates M by (4.4). Since M is Haar by (5.7), it follows by (5.11) that $\dim(M) = r - 1$ and that the ϕ_i's are linearly independent. Now, by (4.5),

$$\begin{aligned} \phi_i(g_1 - g_2) &= \phi_i(-f_i + g_1 + f_i - g_2) \\ &= -r_G(F) + r_G(F) = 0, \qquad i = 1,..., r. \end{aligned} \tag{5.12}$$

But, since $g_1 - g_2 \in M$ and M is a Haar subspace of dimension $r - 1$, it follows that $g_1 - g_2 = 0$ for all pairs of restricted centers in $E_G(F)$. Thus, it follows that $E_G(F)$ is a singleton.

We now extract several corollaries from Theorem 5.1. First, consider the simple case when $I_2 = \varnothing$. This has been studied thoroughly by several authors including Remes [11] and Golomb [6].

COROLLARY 5.1. *Let* $X = C[a, b]$ *and* F *be a compact subset of* X. *Let* $G = g_0 + \operatorname{span}\{v_1, ..., v_n\}$. *If* $\{v_1, ..., v_n\}$ *is a Haar system and* $E_X(F) \cap G = \varnothing$, *then* $E_G(F)$ *is a singleton.*

Following Karlin and Studden [9, p. 25] we call $\{v_1, ..., v_n\}$ a Descartes system if for each subset I of $\{1, ..., n\}$, the set $\{v_i\}_{i \in I}$ is a Haar system.

COROLLARY 5.2. *Let* $X = C[a, b]$, F *be a compact subset of* X, *and* G *be as in* (5.1). *If* $\{v_i\}_{i \in I_2 \cup I_3}$ *is a Descartes system and* $E_X(F) \cap G = \varnothing$, *then* $E_G(F)$ *is a singleton.*

As an application, we note that if $0 < a < b$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, then $\{t^{\alpha_i}\}_{i=1}^n$ is a Descartes system [9, p. 9]. Thus, from Corollary 5.2 we obtain

COROLLARY 5.3. *Let* $X = C[a, b]$, $0 < a < b$, F *be a compact subset of* X, *and* G *as in* (5.1). *If* $v_i = t^{\alpha_i}$ *for* $i \in I_2 \cup I_3$ *and* $E_X(F) \cap G = \varnothing$, *then* $E_G(F)$ *is a singleton.*

Remark 1. A version of the above corollary was proved by Roulier and Taylor [12] in the case of best approximation by polynomials with restricted coefficients. A more general problem of best appropriation is studied by Chalmers [3].

Remark 2. Theorem (5.1) could be generalized by replacing Haar sets by interpolating subspaces of a normed linear space X. For more information on interpolating subspace see [1].

6. RESTRICTED COEFFICIENTS IN $C^1[a, b]$

Let $[a, b]$ be a finite interval and X be the set of continuous functions on $[a, b]$ supplied with the L^1 norm; i.e., $\| z \| = \int_a^b | z(t)| \, dt$. We call this normed linear space $C^1[a, b]$.

LEMMA 6.1. *Let* K *be a convex set in* $X = C^1[a, b]$ *and* F *a bounded set in* X. *Suppose* g_1 *and* g_2 *are in* $E_K(F)$. *Let* $g_0 = (g_1 + g_2)/2$. *Then, for each* $f \in F$ *with* $\| g_0 - f \| = r_K(F)$, *we have for all* t *satisfying* $g_0(t) = f(t)$

$$g_1(t) = g_2(t). \tag{6.1}$$

Proof. Since $E_K(F)$ is convex it follows that $g_0 \in E_K(F)$. Let $f \in F$ be as above, then

$$2 \| g_0 - f \| = \| f - g_1 \| + \| f - g_2 \|, \tag{6.2}$$

since

$$2r_K(F) = 2\,\|g_0 - f\| \leqslant \|g_1 - f\| + \|g_2 - f\| \leqslant 2r_K(F).$$

That is,

$$0 = \int_a^b \{2\,|g_0 - f| - |g_1 - f| - |g_2 - f|\}\,dt. \tag{6.3}$$

But pointwise,

$$2\,|g_0 - f|\,(t) \leqslant |g_1 - f|\,(t) + |g_2 - f|\,(t); \tag{6.4}$$

hence, the integrand must be identically zero by continuity. Thus, if $(g_0 - f)(t) = 0$, it follows that $(g_1 - f)(t) = 0 = (g_2 - f)(t)$, which is equivalent to (6.1).

Recalling the notation in the previous section we may state Theorem 6.1.

THEOREM 6.1. *Let $X = C^1[a, b]$, F a compact subset of X, and G be defined as in* (5.1). *Suppose that for every $J \subset I_2$,*

$$\{v_i\}_{i\in J\cup I_3} \text{ is a Haar system.} \tag{6.5}$$

If $E_X(F) \cap G = \varnothing$, then $E_G(F)$ is a singleton.

Proof. By Theorem (4.1), we obtain r functionals and r associated elements of F and s elements of H satisfying (4.3)–(4.5). Let J_1 be as in (5.8). Then, for any two elements g_1 and g_2 in $E_G(F)$, we must have

$$g_1 - g_2 = \sum_{i\in L} \alpha_i v_i\,, \qquad L = (I_2\backslash J_1) \cup I_3\,. \tag{6.6}$$

As in the proof of Theorem 5.1, it follows that

$$\sum_{i=1}^{r} \lambda_i \phi_i \neq \theta, \tag{6.7}$$

since $E_X(F) \cap G = \varnothing$. Let $g_0 = (g_1 + g_2)/2$. By Lemma 6.1, we have

$$\bigcup_{i=1}^{r} Z(g_0 - f_i) \subset Z(g_1 - g_2), \tag{6.8}$$

where $Z(f)$ denotes the zeroes of a function $f \in C^1[a, b]$. Therefore, since $M = \operatorname{span}\{v_i\}_{i\in L}$ is a Haar set, g_1 can be distinct from g_2 only if the cardinality of $\bigcup_{i=1}^{r} Z(g_0 - f_i)$ is smaller than the cardinality of L. Let us denote the dimension of M by m, which is the cardinality of L. If g_1 is distinct from g_2, then $g_0 - f_i$ must have fewer than m zeroes. Hence, we may assume that the corresponding $\phi_i = \operatorname{sgn}(g_0 - f_i)$. But by (4.4), $\sum_{i=1}^{r} \lambda_i \operatorname{sgn}(g_0 - f_i)$ annihilates M. Since M is a Haar subspace of dimension m it follows that $\sum_{i=1}^{r} \lambda_i \operatorname{sgn}(g_0 - f_i)$ must change sign at least $m+1$ times. Each sign change clearly induces a zero in at least one of the $(g_0 - f_i)$. Therefore, the cardinality of $\bigcup_{i=1}^{r} Z(g_0 - f_i)$ is larger than m and it follows that $g_1 - g_2$ has at least m zeroes. But M is an m-dimensional Haar subspace so g_1 must be g_2. This implies that $E_G(F)$ is a singleton since g_1 and g_2 were arbitrary elements of $E_G(F)$.

Remark 1. We note that $C^1[a, b]$ has no interpolating subspaces, so that the results of Section 5 do not apply.

Remark 2. Since in the course of the proof of Theorem 6.1 we counted sign changes instead of zeroes, we could have weakened the hypothesis (6.5) to "Haar on the open internal (a, b)." Thus, for instance, if $[a, b] = [0, b]$ and $\{v_i\}_{i \in I_2 \cup I_3} = \{1, t, ..., t^k\}$, then with G defined as in (5.1) we have that $E_G(F)$ is a singleton if $E_{C^1(a,b)}(F) \cap G = \varnothing$. Contrast this to the case where $X = C[a, b]$ and G as above, then, even best approximation from G will not necessarly be unique, see [12].

Remark 3. To the best of our knowledge, the result in Theorem 6.1 is new even in the case of best approximation. Of course, results corresponding to Corollaries 5.1 and 5.2 may be drawn in the case of $C^1[a, b]$ via Theorem6.1.

In [2], Carroll obtained the following theorem

THEOREM 6.2. *Let V be a Haar subspace of $C^1[a, b]$ and F a compact subset of $C^1[a, b]$. Then, $E_V(F)$ is a singleton if there is a $g_0 \in E_V(F)$ and a $t \in [a, b]$ so that*

$$(\sup_{f \in F} f(t) - g_0(t))(\inf_{f \in F} f(t) - g_0(t)) > 0. \tag{6.9}$$

Now, we show that Theorem 6.1 contains this result for equicontinuous subsets F. Indeed, if (6.9) obtains we will show that $E_X(F) \cap V = \varnothing$. Suppose that g_0 is as above and that $\inf_{f \in F} f(t) - g_0(t) > 0$ (if $\sup_{f \in F} f(t) - g_0(t) < 0$ a similar argument applies). Let $f(y)$ be the continuous function defined by

$$f(y) = \inf_{f \in F} f(y). \tag{6.10}$$

Then, since g_0 and s are continuous there is a neighborhood (α, β) containing t so that

$$g_0(y) < s(y), \qquad y \in (\alpha, \beta). \tag{6.11}$$

Define $Z = \max(g_0, s)$. Then clearly, for any $f \in F$, z satisfies

$$\begin{aligned} \| g_0 - f \| &= \int_a^b | g_0(y) - f(y) | \, dy \\ &= \int_a^b | g_0(y) - z(y) | + | z(y) - f(y) | \, dy \qquad (6.12) \\ &= \| g_0 - z \| + \| z - f \|. \end{aligned}$$

By (6.11), we have $\| g_0 - z \| \neq 0$, so $r_V(F) > r_X(F)$. Thus, $E_X(F) \cap V = \varnothing$ since $g_0 \in E_V(F)$. We remark that if F is $C^1[a, b]$ compact, but not equicontinuous, then it is an open question as to whether (6.9) implies that $E_X(F) \cap V = \varnothing$.

It is easy to see that condition (6.9) may fail and yet $E_X(f) \cap V = \varnothing$. Consider $[a, b] = [0, 1]$, $V = G = \text{span}\,[1]$, and $F = \{f_1(y) \equiv 0, f_2(y) \equiv \frac{2}{3}, f_3(y) \equiv y\}$. Then, it is easy to see that $g_0(y) \equiv \frac{1}{2} \in E_V(F)$, but $E_X(F) \cap V = \varnothing$ and condition (6.9) does not hold even though we have uniqueness by Theorem 6.1.

7. Examples and Open Questions

In general, we feel that the criterion $E_X(F) \cap G = \varnothing$ is well suited for applications. In the case $C[a, b]$, Kadets and Zamyatin [8] have a very simple characterization for $E_X(F)$, even when F is bounded. In any case, one need only exhibit an element $x_0 \in X$ so that $E_G(F) > \sup_{f \in F} \| x_0 - f \|$.

A natural question that arises is whether in Theorems 5.1 and 6.1 the condition F compact may be weakened to F bounded. Carroll [2] has exhibited a precompact set in $C^1[a, b]$ for which $E_X(F) \cap G = \varnothing$, but $E_G(F)$ is not a singleton. On the other hand, we note that for Haar subspaces of $C[a, b]$, we may weaken the hypothesis to F bounded; see, e.g., [6, 11]. The question is still open in the case of restricted coefficients.

In $C[a, b]$ the Haar condition on the v_i's can clearly not be weakened since a finite-dimensional subspace of $C[a, b]$ is Chebyshev if and only if it is Haar. In $C^1[a, b]$, we have seen that we may weaken Haar to Haar on (a, b). A question that remains is whether Theorem 6.1 is true with "Haar" replaced by "$\text{span}\{v_i\}_{i \in J \cup I_3}$, $J \subset I_2$ is Chebyshev." See Example 2 below. This question can be asked for general normed spaces. However, we have the following two counterexamples. One is a finite-dimensional Chebyshev subspace of $l^1(3)$ and the other an infinite-dimensional Chebyshev (resp. semi-Chebyshev) subspace of $L^1[0, 1]$ (resp. $C^1[0, 1]$).

EXAMPLE 1. Let $X = l^1(3)$, $G = \text{span}\{(1, 0, 0), (0, 1, 0)\}$, and $F = \{(1, 1, 1,), (-1, -1, 1)\}$. Then, $r_X(F) = 2$ and $r_G(F) = 3$, but both $\theta = (0, 0, 0)$ and $g = (1, -1, 0)$ are restricted centers.

EXAMPLE 2. Let $X = L^1[0, 1]$, $G = \{g\colon g(t) = 0 \text{ a.e. } t > \frac{1}{2}\}$ and $F = \{t - \frac{1}{2}, |t - \frac{1}{2}|\}$. Then, $E_X(F) \cap G = \varnothing$. Furhtermore, if $g \in G$ satisfies $t - \frac{1}{2} \leqslant g(t) \leqslant |t - \frac{1}{2}|$ for $0 \leqslant t \leqslant \frac{1}{2}$ and if $\int_0^1 g(t)\,dt = 0$, then $g \in E_G(F)$ and hence, $E_G(F)$ is not a singleton. This example also provides us with a semi-Chebyshev subspace G_1 of $C^1[0, 1]$, e.g., $G \cap C^1[0, 1]$, for which $E_{G_1}(F)$ has more than one element even though $E_X(F) \cap G_1 = \varnothing$.

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